# THE PROPAGATION OF A SPHERICAL BLAST WAVE IN A HEAT-CONDUCTING GAS

(RASPROSTRANENIE SFERICHESKOI VZRYVNOI VOLNY V TEPLOPROVODNOM GAZE)

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The problem of the propagation of a blast wave in a heat-conducting gas with spherical symmetry is considered in [1,2]. At the initial instant $t = 0$ there is an instantaneous release of a finite amount of energy $E_0$ at the point $r = 0$ in gas which is at rest. The problem is self-similar [3] if we neglect the initial pressure $p_0$ and the coefficient of heat conductivity is taken equal to $\kappa_1 T^{1/6}$ (where $\kappa_1$ is a constant and $T$ is the temperature). In the papers cited above a solution is constructed by using the Hugoniot conditions at the shock under the assumption that the temperature undergoes a discontinuity.

It is demonstrated below that the assumption of discontinuity of temperature makes the problem many-valued, whereas with continuity of temperature the solution is obtained uniquely. Moreover there exists a constant $A_*$ which divides all solutions of the problem into two types, and numerical solutions of the problem are obtained for both cases.

We notice that the problem of blast in a heat-conducting gas with cylindrical symmetry under the assumption of continuity of temperature at the shock is solved in [4].

**1.** The system of equations of gas-dynamics in Eulerian coordinates has the form

$$\rho\left(\frac{\partial u}{\partial t}+u\,\frac{\partial u}{\partial r}\right)+R\,\frac{\partial(\rho T)}{\partial r}=0,\qquad \frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial r}+\frac{2\rho u}{r}=0 \tag{1.1}$$

$$\frac{\partial}{\partial t}\left[\rho r^2\left(\frac{u^2}{2}+\frac{RT}{\gamma-1}\right)\right]+\frac{\partial}{\partial r}\,r^2\left[\rho u\left(\frac{u^2}{2}+\frac{RT}{\gamma-1}\right)+R\rho Tu-\varkappa_1 T^{1/6}\,\frac{\partial T}{\partial r}\right]=0$$

Here $u$ is velocity, $\rho$ is density, $R$ is the gas constant, $r$ is the radius and $\gamma$ is Poisson's adiabatic exponent. The boundary conditions are

at the center

$$u\,(0,\ t) = 0 \tag{1.2}$$

at the front of the disturbance $r_0(t)$

$$u\,(r_0,\ t) = 0, \qquad \rho\,(r_0,\ t) = \rho_0, \qquad T\,(r_0,\ t) = 0 \tag{1.3}$$

at the shock $r_1(t)$

$$\rho_1\,(u_1 - c) = \rho_2\,(u_2 - c), \quad \rho_1\,(u_1 - c)^2 + R\rho_1 T_1 = \rho_2\,(u_2 - c)^2 + R\rho_2 T_1 \tag{1.4}$$

$$\rho_1\,(u_1 - c)\left(\frac{u_1^2}{2} + \frac{RT_1}{\gamma - 1}\right) + R\rho_1 T_1 u_1 - \varkappa_1 T_1^{1/6}\left(\frac{\partial T}{\partial r}\right)_1 =$$

$$= \rho_2\,(u_2 - c)\left(\frac{u_2^2}{2} + \frac{RT_1}{\gamma - 1}\right) + R\rho_2 T_1 u_2 - \varkappa_1 T_1^{1/6}\left(\frac{\partial T}{\partial r}\right)_2$$

Here $c$ is the shock velocity, the index 1 refers to quantities before the shock front and the index 2 to quantities behind the shock front. Moreover, the required functions must satisfy the equation

$$E_0 = 4\pi \int_0^{r_0} \left(\frac{\rho u^2}{2} + \frac{R\rho T}{\gamma - 1}\right) r^2 dr \tag{1.5}$$

Since the problem is self-similar, we have

$$r_0 = \left(\frac{\alpha E_0}{\rho_0}\right)^{1/5} t^{2/5}, \qquad r_1 = \lambda_1 r_0, \qquad c_0 = \frac{2}{5}\,\frac{r_0}{t}, \qquad c = \lambda_1 c_0$$

$$u = \frac{2c_0}{\gamma + 1}\,f\,(\lambda), \qquad \rho = \frac{\gamma + 1}{\gamma - 1}\,\rho_0 g\,(\lambda), \qquad T = \frac{2c_0^2\,(\gamma - 1)}{R\,(\gamma + 1)^2}\,\Theta\,(\lambda), \qquad \lambda = \frac{r}{r_0} \tag{1.6}$$

Here $\lambda$, $\alpha$ are dimensionless constants, determined during the solution of the problem, $c_0$ is the velocity of propagation of the disturbance.

After substituting (1.6) in equations (1.1) and (1.2) to (1.5) we obtain the following system of ordinary differential equations

$$\left(f - \frac{\gamma + 1}{2}\,\lambda\right) f' - \frac{3\,(\gamma + 1)}{4}\,f + \frac{\gamma - 1}{2g}\,(g\Theta)' = 0, \qquad \left(f - \frac{\gamma + 1}{2}\,\lambda\right) g' + g\left(f' + \frac{2f}{\lambda}\right) = 0$$

$$A\Theta^{1/6}\Theta' - \frac{12\,(\gamma - 1)}{\gamma + 1}\,g\Theta f + 6g\left(\lambda - \frac{2}{\gamma + 1}\,f\right)(f^2 + \Theta) = 0 \tag{1.7}$$

The dimensionless constant $A$ is given by

$$A = \frac{6 \cdot 2^{1/6}\,(\gamma - 1)^{13/6}\,(\gamma + 1)^{-4/6}\varkappa_1}{R^{7/6}\,(0.4\rho_0)^{2/3}\,(\alpha E_0)^{1/3}}$$

In deriving the last equation of the system (1.7) we have used the following integral, found in [3]:

$$r^2 \left[\left(\frac{2}{5}\,\frac{r}{t} - u\right)\left(\frac{\rho u^2}{2} + \frac{R\rho T}{\gamma - 1}\right) - R\rho T u + \varkappa_1 T^{1/6}\,\frac{\partial T}{\partial r}\right] = C$$

The constant of integration is chosen from the conditions at the front of the disturbance (1.3).

The boundary conditions (1.2) to (1.4) and expression (1.5) for the dimensionless functions take the form:

$$f(1) = 0, \qquad g(1) = \frac{\gamma - 1}{\gamma + 1}, \qquad \Theta(1) = 0 \tag{1.8}$$

$$f(0) = 0 \tag{1.9}$$

$$\begin{gathered} g_1\left(\frac{2}{\gamma+1}\,f_1 - \lambda_1\right) = g_2\left(\frac{2}{\gamma+1}\,f_2 - \lambda_1\right) \\ g_1\left(\frac{2}{\gamma+1}\,f_1 - \lambda_1\right)^2 + \frac{2(\gamma-1)}{(\gamma+1)^2}\,g_1\Theta_1 = g_2\left(\frac{2}{\gamma+1}\,f_2 - \lambda_1\right)^2 + \frac{2(\gamma-1)}{(\gamma+1)^2}\,g_2\Theta_1 \\ 6g_1\left(\frac{2}{\gamma+1}\,f_1 - \lambda_1\right)(f_1^2 + \Theta_1) + \frac{12(\gamma-1)}{\gamma+1}\,g_1\Theta_1 f_1 - A\Theta_1^{1/6}\Theta_1' = \\ = 6g_2\left(\frac{2}{\gamma+1}\,f_2 - \lambda_1\right)(f_2^2 + \Theta_1) + \frac{12(\gamma-1)}{\gamma+1}\,g_2\Theta_1 f_2 - A\Theta_1^{1/6}\Theta_2' \end{gathered} \tag{1.10}$$

$$\frac{1}{\alpha} = \frac{8\pi(0.4)^2}{(\gamma+1)(\gamma-1)}\int_0^1 g\,(f^2 + \Theta)\,\lambda^2 d\lambda \tag{1.11}$$

The problem is reduced to integration of the system of ordinary differential equations (1.7) with the boundary conditions (1.8) and (1.9).

**2.** Numerical computation shows that there does not exist a continuous solution satisfying the conditions (1.8) and (1.9). The front of the disturbance is determined by the point

$$I\,(\lambda = 1, \quad \Theta = 0, \quad f = 0, \quad g = (\gamma - 1)/(\gamma + 1))$$

which is a singular point for the system (1.7).

In the neighborhood of the point $I$ there exists an approximate representation of the solution which is independent of arbitrary constants and has the form

$$\begin{gathered} \Theta = \left[\frac{\gamma-1}{2(\gamma+1)A}\right]^6 (1-\lambda^2)^6, \qquad g = \frac{\gamma-1}{\gamma+1}\exp\int_\lambda^1 \frac{g\,(\lambda f' + 2f)}{(f - {}^1/_2(\gamma+1)\lambda)}\,d\lambda \\ f = 12\,\frac{\gamma-1}{\gamma+1}\left[\frac{\gamma-1}{2(\gamma+1)A}\right]^6 \lambda^{-3/2}\int_\lambda^1 (1-\lambda^2)^5\,\lambda^{3/2} d\lambda \end{gathered} \tag{2.1}$$

Making use of this expansion, we can try to construct a continuous solution. But while integrating numerically towards the center we encounter the point $\lambda^0$, at which $f'(\lambda^0) = \infty$ (Fig. 1). Accordingly, the continuous construction of a solution is impossible and at a certain point $\lambda_1$ we must assume a discontinuity ($\lambda^0 < \lambda_1 < 1$).

Solutions incorporating a discontinuity can be sought in two ways: assuming that the temperature changes continuously at the shock, or assuming a discontinuity. The latter assumption makes the solution of the problem multi-valued, since together with the arbitrary parameter $\lambda_1$ yet another parameter makes its appearance: at the temperature discontinuity the three Hugoniot conditions connect the four unknown quantities $\Theta_2$, $f_2$, $g_2$, $\Theta_2'$. The quantities before the front are known from numerical integration of the system (1.7). Accordingly, for each fixed value of $\lambda_1$, by variation of the second parameter we can satisfy the condition at the center $f(0) = 0$ and obtain a solution satisfying the postulated problem. In [1] the solution is found for $\lambda_1 = 1$. However, by changing $\lambda_1$ we can obtain an infinite number of solutions of one and the same problem, which according to physical sense has a unique solution.

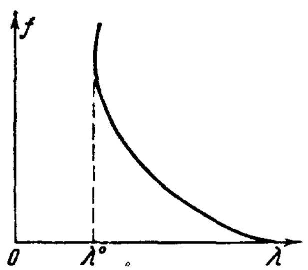


Fig. 1.

If the temperature at the shock is assumed to be continuous, then there remains one arbitrary parameter $\lambda_1$, determined from the conditions at the center $f(0) = 0$. The solution in this case is obtained uniquely. Below it will be shown that solutions with discontinuous temperature are unstable and pass into the solution with continuous temperature.

Accordingly, we shall assume the temperature to be continuous and take Hugoniot's conditions in the form (1.10). The solution must leave the point $I$ and satisfy the condition at the center, i.e. it must pass through the point

$$II \ (\lambda = 0, f = 0, \Theta = \Theta_0, g = g_0)$$

where $\Theta_0$ and $g_0$ are as yet undetermined constants. The center is a singular point of the system (1.7), and in its neighborhood the following expansion [1] is valid

$$f = \frac{3}{5}\,\frac{(\gamma+1)g_0}{A\Theta_0^{1/4}}\,\lambda^3, \qquad \Theta = \Theta_0 - 3\,\frac{g_0\Theta_0^{3/4}}{A}\,\lambda^2, \qquad g = g_0 + 3\,\frac{g_0^2}{A\Theta_0^{1/4}}\lambda^2 \qquad (2.2)$$

For numerical integration of the system (1.7) it is convenient to present it in the form

$$\Theta' = \frac{12g}{(\gamma+1)A\Theta^{1/6}}\left((\gamma-1)\Theta f + \left[f - \frac{\gamma+1}{2}\lambda\right](f^2+\Theta)\right) \qquad (2.3)$$

$$f' = \left(\left[\frac{3(\gamma+1)}{4}f - \frac{\gamma-1}{2}\Theta'\right]\left[f - \frac{\gamma+1}{2}\lambda\right] + \frac{\gamma-1}{\lambda}\Theta f\right)\left(\left[f - \frac{\gamma+1}{2}\lambda\right]^2 - \frac{\gamma-1}{2}\Theta\right)^{-1}$$

$$g' = -\left(g\left[f' + 2\frac{f}{\lambda}\right]\left(f - \frac{\gamma+1}{2}\lambda\right)^{-1}\right.$$

The behavior of the integral curves close to the center is approximately described by the following expansion, which is obtained from system (2.3) if in the right-hand sides of it we neglect terms of a higher order of smallness

$$f = c\lambda^{-2} + \frac{3}{5}\frac{(\gamma+1)g_0}{A\Theta_0^{1/6}}\lambda^3 + \ldots, \qquad \Theta = \Theta_0 + \ldots, \qquad g = g_0 + \ldots \qquad (2.4)$$

In numerical integration of the system (2.3) in the direction from the point

$$III\ (\lambda_1,\ \Theta_1,\ f_2,\ g_2)$$

to the point *II* the integral curves spread out so quickly that in practice it becomes impossible to integrate in this direction because of rounding-off errors. In order to make the calculation stable it is necessary for the second equation of the system (2.3) to be integrated in the stable direction - from point *II* to point *III* - and the other two equations from point *III* to point *II*. In this case, for solution of the two-point boundary problem with $A \leqslant A_* > 0$ (the value of $A_*$ is determined below) we can construct, just as in [7], a convergent iteration process. Suppose that $\lambda_1$ is fixed; then from the conditions (1.10) we can find $f_2$ and $g_2$ in terms of the known quantities $\lambda_1$, $\Theta_1$, $f_1$ and $g_1$

$$f_2 = \frac{\gamma+1}{2}\left[\lambda_1 - \frac{2(\gamma-1)}{(\gamma+1)^2}\frac{\Theta_1}{\lambda_1 - 2f_1/(\gamma+1)}\right], \quad g_2 = \frac{(\gamma+1)^2}{2(\gamma-1)\Theta_1}\left(\lambda_1 - \frac{2}{\gamma+1}f_1\right)^2 g_1 \qquad (2.5)$$

Let us take $\lambda = \lambda_0$, lying close to zero, so that to a sufficient degree of accuracy we can use the expansion (2.2). Let us assume the function $f$ in the interval $[\lambda_0, \lambda_1]$ after which, knowing $\Theta_1$ and $g_2$ when $\lambda = \lambda_1$, we integrate the first and third equations of the system (2.3) and determine $\Theta(0)$ and $g(0)$. Starting from $\lambda = \lambda_0$ according to the expansion (2.2), and solving the second equation of the system (2.3), we can improve $f$, and so on. The process described is reiterated until such time as the required accuracy is attained. As a result we obtain a certain $f_2^0$ when $\lambda = \lambda_1$. By means of trial and error $\lambda_1$ is chosen so that $f_2^0 = f_2$.

**3.** When $A > A_*$ the numerical solution of the problem is appreciably more complicated. Between the points *II* and *III* there occur two more singular points, through which the required solution passes. For an explanation of the nature of the singularities arising, let us consider the system (2.3) when $A = \infty$. Let us consider the second equation of the system

$$f' = \left(\frac{3(\gamma+1)}{4} f\left[f - \frac{\gamma+1}{2}\lambda\right] + \frac{(\gamma-1)}{\lambda}\theta_2 f\right)\left(\left[f - \frac{\gamma+1}{2}\lambda\right]^2 - \frac{\gamma-1}{2}\theta_2\right)^{-1} \tag{3.1}$$

A similar equation is considered in [5]. When $\lambda \geqslant 0$ it has three singular points:

saddle

$$D(\lambda = 0,\ f = 0)$$

node

$$B\left(\lambda = \frac{2}{\gamma+1}\left(\frac{\gamma-1}{2}\theta_2\right)^{1/2},\ \ f = 0\right)$$

saddle

$$C\left(\lambda = \frac{8}{3(\gamma+1)}\left(\frac{\gamma-1}{2}\theta_2\right)^{1/2},\ \ f = \frac{1}{3}\left(\frac{\gamma-1}{2}\theta_2\right)\right)$$

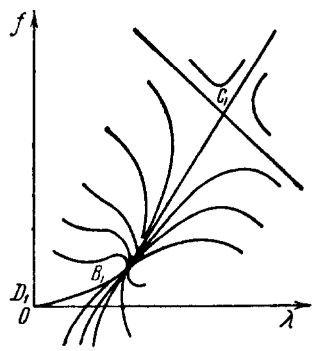


Fig. 2.

In the general case, when $A \neq \infty$, $A > A_*$, the coordinates of the singular points and the nature of the behavior of the integral curves are determined during the process of solution of each actual problem.

Numerical solution was carried out for $\gamma = 1.4$. It turned out that the borderline constant $A_*$ is included between the limits $0.0833 < A_* < 0.1666$. When $A \leqslant A_*$ a solution of the first type occurs. It is found by the iterative method described in Section 2. If $A > A_*$, then numerical integration is carried out by the same iterative method, but complicated by determination of the function $f$, since for the $\theta$ and $g$ found on each iteration the second equation of the system (2.3) will have two more singular points $B_1$ and $C_1$ between the points $D_1$ and $(\lambda_1,\ f_2^{\circ})$. For $A = 0.1666$ the qualitative behavior of the integral curves agrees with that obtained when $A = \infty$, i.e. the point $B_1$ is a node and $C_1$ is a saddle (Fig. 2). Numerical computations show that as $A$ increases the nature of the singular points $B_1$ and $C_1$ does not change, and already when $A > 1$ we can take $\theta'(\lambda) = 0 (0 \leqslant \lambda \leqslant \lambda_1)$ and make use of equation (3.1).

Integration of the equation for $f$ must always be taken in the stable direction, taking account of the behavior of the integral curves: from

$C_1$ to $B_1$, from $C_1$ to $(\lambda_1, f_2^{\circ})$, from $D_1$ to $B_1$ and at the points $B_1$ and $C_1$ making use of the following expansion for $f$

$$f = f_k + f'_k(\lambda - \lambda_k), \qquad f'_k = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 2\left(f_k - \frac{\gamma+1}{2}\lambda_k\right), \qquad b = \frac{7}{8}(\gamma+1)a + \frac{3}{4}(\gamma+1)f_k + \frac{\gamma-1}{\lambda_k}\theta_k$$

$$c = \frac{\gamma-1}{4}\theta''_k a + \frac{\gamma+1}{2}\left[\frac{3(\gamma+1)}{4}f_k - \frac{\gamma-1}{2}\theta'_k\right] - (\gamma-1)\left(\theta'_k - \frac{\theta_k}{\lambda_k}\right)\frac{f_k}{\lambda_k}$$

where $\lambda_k$, $f_k$, $\theta_k$ are referred to the respective singular point.

**4.** Accordingly, a blast in a heat-conducting gas with the coefficient of thermal conductivity $\kappa = \kappa_1 T^{1/6}$ is described by a self-similar solution with continuity of temperature. The density $\rho$ and the velocity $u$ undergo discontinuities satisfying the Hugoniot conditions with continuity of temperature. The disturbance in the spherical case, in contrast to the cylindrical case, when $\kappa = \kappa_1$, propagates with finite velocity.

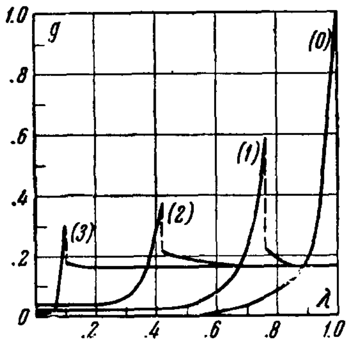

Fig. 3.

The problem with $\gamma = 1.4$ has solutions of two types: the solution of the first type, when there are no singular points between the center and the shock, and all quantities vary smoothly; and the solution of the second type, when there are singular points between the center and the shock, and the required solution passes through them. At a point lying close to the center the functions $f(\lambda)$ and $g(\lambda)$ undergo a jump. This point corresponds to a weak discontinuity. Through the other point, which is a saddle, the solution passes in a smooth fashion.

The type of solution is determined by the value of the dimensionless constant $A$. If $\gamma = 1.4$, then the value of the borderline constant $A_*$ is included between the limits $0.0833 < A_* < 0.1666$. When $A < A_*$ the solution is of the first type, whilst if $A > A_*$ the solution is of the second type. Numerical integration of the problem was carried out for the following values: $A$ = (1) 0.0417, (2) 0.0833, (3) 0.1666, (4) 1.048. In Figs. 3 to 5 are shown the curves for (0-3), the symbol (0) denoting the solution of Sedov, corresponding to $A = 0$. The values of the dimensionless quantity $\alpha$ are respectively equal to:

$(0)\ 0.117\cdot 10$, $(1)\ 0.760\cdot 10$, $(2)\ 0.789\cdot 10^2$, $(3)\ 0.400\cdot 10^4$, $(4)\ 0.240\cdot 10^9$

If we fix the initial parameters $\gamma$, $R$, $E_0$, $\rho_0$ and vary only the constant $\kappa_1$, then correspondingly the quantity $A$ will also change, being

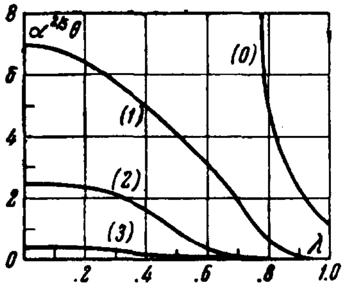

Fig. 4.

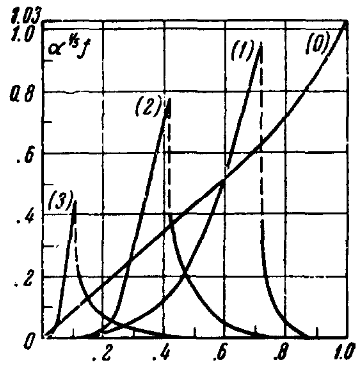

Fig. 5.

linearly dependent on $\kappa_1$. Suppose that $\kappa_1 \to 0$, then $A \to 0$ also. This means that the influence of heat conduction disappears and the solution must tend in the limit to the well known solution of Sedov's problem on blast in gas without allowance for heat conduction [6]. In fact, from the graphs presented it follows that for decreasing $A$ the quantity $\lambda_1$ tends to unity and the solution with continuity of temperature tends to the solution with discontinuous temperature.

For large values of $A$ and correspondingly large values of $\kappa$ the influence of thermal conductivity becomes decisive, the point of discontinuity of the functions $f$ and $g$ approaches $\lambda = 0$.

The solution obtained is stable. Various existing methods enable us to find approximately the solution of a wide class of problems, including some which are not self-similar. If the instantaneous release of a finite amount of energy $E_0$ occurs not at the point $r = 0$, but in a certain sphere of radius $R_0$ then the problem is no longer self-similar. But the solution for finite $R_0$ tends with increasing time towards the solution already found. This assertion was verified by a separate calculation on a fast computer. The initial distribution of $u$, $T$, $\rho$ for $0 \leqslant r \leqslant R_0$ was postulated arbitrarily, but taking account of the fact that $E = E_0$. In particular, for an initial profile we could select any solution with discontinuous temperature, e.g. the solution obtained in [1]. In any case the motion under consideration converges to the self-similar regime already obtained. The solutions with discontinuous temperature are unstable and change into the solution with continuity of temperature.

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